Calculus 140, section 3.1 Derivatives

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We include section 3.1 with Chapter 2 on Exam 1 because it really is just a small extension of the topics of Chapter 2.

From sections 2.1 and 2.2, we have that the **slope of a line tangent to a graph at a point where *x* = *a*** is

$$m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Then, substituting into the (I hope) familiar point-slope formula [$y - y_1 = m(x - x_1)$] we get that the **equation of a line tangent to a graph at a point where *x* = *a*** is

$$y - f(a) = m_a(x - a) \text{ or } y = f(a) + m_a(x - a).$$

We're now going to formalize this into the definition of "one of the two central concepts of calculus: the derivative."

Definition 3.1: "Let *a* be a number in the domain of the function *f*. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists, we call this limit the **derivative of *f* at *a***, and denote it by $f'(a)$, so that $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$."

If this limit exists, we'll use terminology such as "*f* has a derivative at *a*" and "*f* is differentiable at *a*".

The (first) derivative of *f* has several notations that we will use on a regular basis: f', $f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx}[f(x)]$.

Others that you might see in other texts include $\dot{u}$, $Df(x)$, $D_x f$. (Note the dot over the *u*.) [See Table 3.1 in the text.]

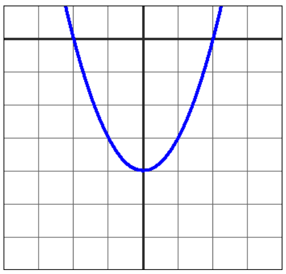
Back to lines tangent to a curve.

By definition, the slope of the line tangent to a curve at the point (*a*, *f*(*a*)) is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

By extension, the equation of the line tangent to to the curve at the point (*a*, *f*(*a*)) is

$$y - f(a) = f'(a)(x - a) \text{ or equivalently } y = f(a) + f'(a)(x - a).$$

Example A: Find the equation of the line tangent to $f(x) = x^2 - 4$ at *x* = 1.



Theorem 3.2: "If f is differentiable at a, then f is continuous at a, that is $\lim_{x \to a} f(x) = f(a)$."

This follows from the definitions of differentiable and continuity. See the text's proof for details.

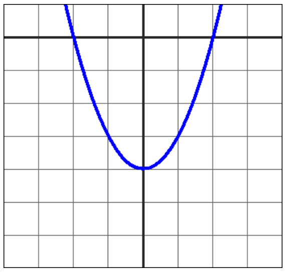
IMPORTANT: This is a one-way conditional statement! While differentiable implies continuity, continuous *does not* imply differentiable. See the text's Example 3 which looks at $f(x) = |x|$ at $x = 0$.

It would be labor-intensive (and impossible) to use the definition above to find the (first) derivative of a given function for every one of the infinite points in its domain.

Instead, we're going to generalize the process to find a formula that can be used for any value x in the domain of a given function.

Specifically, given a differentiable function f, the (first) derivative of f is given by $f'(x) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x}$.

Example A revisited: Given $f(x) = x^2 - 4$, find a formula for the (first) derivative of f, that is, for $f'(x)$. Then, use your formula to find $f'(a)$ for various values a.



Hint for homework: You may find the text's Example 5 $\left(f(x) = \sqrt{x}\right)$ useful when searching for a technique to use for homework questions.

One last note on applications (i.e. word problems). In applications questions, the first derivative of some functions takes on a very specific meaning, one of which the text explores in Example 2, and you will be asked to evaluate in homework exercises:

"velocity is the derivative of the position function: $v(t) = f'(t)$

marginal cost is the derivative of the cost function: $m_C(x) = C'(x)$

marginal revenue is the derivative of the revenue function: $m_R(x) = R'(x)$".